Cavity QED and Quantum Computation in the Weak Coupling Regime II : Complete Construction of the Controlled–Controlled NOT Gate

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Abstract

In this paper we treat a cavity QED quantum computation. Namely, we consider a model of quantum computation based on n atoms of laser–cooled and trapped linearly in a cavity and realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields.

We solve the Schrödinger equation of the model in the weak coupling regime to construct the controlled NOT gate in the case of n=2, and to construct the controlled–controlled NOT gate in the case of n=3 by making use of several resonance conditions and rotating wave approximation associated to them. We also present an idea to construct general quantum circuits.

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The approach is more sophisticated than that of the paper [K. Fujii, Higashida, Kato and Wada, Cavity QED and Quantum Computation in the Weak Coupling Regime, J. Opt. B : Quantum Semiclass. Opt. **6** (2004), 502].

Our method is not heuristic but completely mathematical, and the significant feature is based on a consistent use of Rabi oscillations.

1 Introduction

Quantum Computation (or Computer) is a challenging task in this century for not only physicists but also mathematicians. Quantum Computation is in a usual understanding based on qubits which are based on two level systems (two energy levels or fundamental spins) of atoms, See [1] as for general theory of two level systems. The essence of Quantum Computation is to construct an element of huge unitary group $U(2^n)$ by manipulating n atoms with photons, laser fields, etc.

In a realistic image of Quantum Computer we need at least one hundred atoms. However, we may meet a very severe problem called Decoherence which destroys a superposition of quantum states in the process of unitary evolution of our system through some influence arising from natural environment. At the moment it is not easy to control the decoherence. See for example the papers in [2] as an introduction to the problem.

An optical system like Cavity QED may have some advantage on this problem, so we want to consider a model of quantum computation based on n atoms of laser–cooled and trapped linearly in a cavity. As an approximate model we realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields. As to the Tavis–Cummings model see [4]. To perform the quantum computation we must first of all show that our system is universal [5]. To show it we must construct the controlled NOT operator (gate) explicitly in the case of $n = 2$, [5], [6].

For that we must embed a system of two–qubits into a space of wave functions of the model and solve the Schrödinger equation. In a reduced system we can construct the controlled NOT by use of some resonance condition and the rotating wave approximation

associated to it. Then we need to assume that the coupling constants are small enough (the weak coupling regime in the title).

Next we want to construct the controlled–controlled NOT operator in the case of $n = 3$. For that purpose the construction of controlled NOT gates of three types is required [1] because three atoms are trapped **linearly** in the cavity. We have given an idea to construct them in [3], which is however not complete. In this paper we give a complete construction to the controlled–controlled NOT gate.

However, to push on with our method (namely, for the case of n atoms ($n \geq 5$)) is not easy by some severe technical reason. Therefore we present the idea in [3] once more to construct general quantum circuits, which will give a general quantum computation.

The contents of this paper are as follows :

2 A Model Based on Cavity QED

We consider a quantum computation model based on n atoms of laser–cooled and trapped linearly in a cavity and realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields. This is of course an approximate theory. In a more realistic model we must add other dynamical variables such as positions of atoms and their momenta etc. However, since such a model is almost impossible to solve we consider a simple one.

[1] In the study of Cavity QED Quantum Computation this important point is missed

Then the Hamiltonian is given by

$$H = \omega 1_L \otimes a^\dagger a + \frac{\Delta}{2}\sum_{j=1}^{n}\sigma_j^{(3)} \otimes \mathbf{1} + g\sum_{j=1}^{n}\left(\sigma_j^{(+)} \otimes a + \sigma_j^{(-)} \otimes a^\dagger\right) + \\ \sum_{j=1}^{n} h_j\left(\sigma_j^{(+)}\mathrm{e}^{i(\Omega_j t+\phi_j)} + \sigma_j^{(-)}\mathrm{e}^{-i(\Omega_j t+\phi_j)}\right) \otimes \mathbf{1} \tag{1}$$

where ω is the frequency of radiation field, Δ the energy difference of two level atoms, a and $a^\dagger$ are annihilation and creation operators of the field, and g a coupling constant, Ω_j the frequencies of external fields which are treated as classical fields, h_j coupling constants, and $L = 2^n$. Here $\sigma_j^{(+)}$, $\sigma_j^{(-)}$ and $\sigma_j^{(3)}$ are given as

$$\sigma_j^{(s)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_s \otimes 1_2 \otimes \cdots \otimes 1_2 \ (j-\text{position}) \ \in \ M(L, \mathbf{C}) \tag{2}$$

where s is $+$, $-$ and 3 respectively and

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

See the figure 1 as an image of the model. Note that the cases of $n = 2$ and 3 are our target through this paper. Here we state our scenario of quantum computation. Each independent external field generates a unitary element of the corresponding qubit (atom) like $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ where $a_j \in U(2)$, while a photon inserted generates an entanglement among such elements like $\sum a_1 \otimes a_2 \otimes \cdots \otimes a_n$. As a whole we obtain any element in $U(2^n)$ (a universality).

Here let us rewrite the Hamiltonian (1). If we set

$$S_+ = \sum_{j=1}^{n}\sigma_j^{(+)}, \quad S_- = \sum_{j=1}^{n}\sigma_j^{(-)}, \quad S_3 = \frac{1}{2}\sum_{j=1}^{n}\sigma_j^{(3)}, \tag{4}$$

then (1) can be written as

$$H = \omega 1_L \otimes a^\dagger a + \Delta S_3 \otimes \mathbf{1} + g\left(S_+ \otimes a + S_- \otimes a^\dagger\right) + \\ \sum_{j=1}^{n} h_j\left(\sigma_j^{(+)}\mathrm{e}^{i(\Omega_j t+\phi_j)} + \sigma_j^{(-)}\mathrm{e}^{-i(\Omega_j t+\phi_j)}\right) \otimes \mathbf{1} \equiv H_0 + V(t), \tag{5}$$

which is relatively clear. H_0 is the Tavis–Cummings Hamiltonian and we treat it as an unperturved one. We note that $\{S_+, S_-, S_3\}$ satisfy the $su(2)$–relation

$$[S_3, S_+] = S_+, \quad [S_3, S_-] = -S_-, \quad [S_+, S_-] = 2S_3. \tag{6}$$

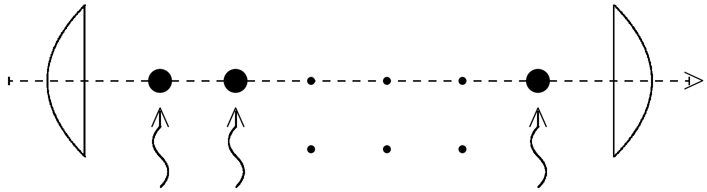


Figure 1: The general setting for a quantum computation based on Cavity QED. The dotted line means a single photon inserted in the cavity and all curves mean external (laser) fields (which are treated as classical ones) subjected to atoms

However, the representation ρ defined by

$$\rho(\sigma_+) = S_+, \quad \rho(\sigma_-) = S_-, \quad \rho(\sigma_3/2) = S_3$$

is a full representation of $su(2)$, which is of course not irreducible.

We would like to solve the Schrödinger equation

$$i\frac{d}{dt}U = HU = (H_0 + V)\,U, \tag{7}$$

where U is a unitary operator. As an equivalent form let us change to the interaction picture, which is performed by the **method of constant variation**. The equation $i\frac{d}{dt}U = H_0 U$ is solved to be

$$U(t) = \left(\mathrm{e}^{-it\omega S_3} \otimes \mathrm{e}^{-it\omega N}\right) \mathrm{e}^{-itg\left(S_+\otimes a + S_-\otimes a^\dagger\right)} U_0$$

where $N = a^\dagger a$ is the number operator and U_0 a constant unitary. Here we have used the resonance condition

$$\omega = \Delta \tag{8}$$

, see for example [7]. By changing $U_0 \longmapsto U_0(t)$ and substituting into (7) we obtain the equation

$$i\frac{d}{dt}U_0 = \mathrm{e}^{itg\left(S_+\otimes a + S_-\otimes a^\dagger\right)} \left(\mathrm{e}^{it\omega S_3} \otimes \mathrm{e}^{it\omega N}\right) V(t) \left(\mathrm{e}^{-it\omega S_3} \otimes \mathrm{e}^{-it\omega N}\right) \mathrm{e}^{-itg\left(S_+\otimes a + S_-\otimes a^\dagger\right)} U_0 \tag{9}$$

after some algebras. This is the interaction picture of (7) and we use this for our quantum computation. Therefore we must calculate the right hand side of (9) explicitly, which is

however a very hard task due to the (complicated) term $\mathrm{e}^{-itg\left(S_+\otimes a+S_-\otimes a^\dagger\right)}$. For convenience in the following we set

$$A \equiv A_n = S_+ \otimes a + S_- \otimes a^\dagger. \tag{10}$$

It has been done only for $n = 1$, 2, 3 and 4 as far as we know, see [7], [8]. We list the calculations for $n = 1$, 2, 3 in the following, [7].

One Atom Case In this case A in (10) is written as

$$A_1 = \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \equiv B_{1/2}. \tag{11}$$

By making use of the simple relation

$$A_1^2 = \begin{pmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{pmatrix} = \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} \tag{12}$$

we have

$$\begin{aligned}
\mathrm{e}^{-itgB_{1/2}} = \mathrm{e}^{-itgA_1} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (tg)^{2n} A_1^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (tg)^{2n+1} A_1^{2n+1} \\
&= \begin{pmatrix} \cos\left(tg\sqrt{N+1}\right) & -i\frac{\sin\left(tg\sqrt{N+1}\right)}{\sqrt{N+1}}a \\ -i\frac{\sin\left(tg\sqrt{N}\right)}{\sqrt{N}}a^\dagger & \cos\left(tg\sqrt{N}\right) \end{pmatrix} \\
&\equiv \begin{pmatrix} C(N+1) & -iS(N+1)a \\ -iS(N)a^\dagger & C(N) \end{pmatrix}.
\end{aligned} \tag{13}$$

We obtained the explicit form of solution. However, this form is more or less well–known, see for example the second book in [1].

Two Atoms Case In this case A in (10) is written as

$$A_2 = \begin{pmatrix} 0 & a & a & 0 \\ a^\dagger & 0 & 0 & a \\ a^\dagger & 0 & 0 & a \\ 0 & a^\dagger & a^\dagger & 0 \end{pmatrix}. \tag{14}$$

Our method is to reduce the 4×4–matrix A_2 in (14) to a 3×3–matrix B_1 in the following to make our calculation easier. For that aim we prepare the following matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

then it is easy to see

$$T^{\dagger} A_2 T = \begin{pmatrix} 0 & & & \\ & 0 & \sqrt{2}a & 0 \\ & \sqrt{2}a^{\dagger} & 0 & \sqrt{2}a \\ & 0 & \sqrt{2}a^{\dagger} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \\ & B_1 \end{pmatrix}$$

where $B_1 = J_+ \otimes a + J_- \otimes a^{\dagger}$ and $\{J_+, J_-\}$ are just generators of (spin one) irreducible representation of (3). We note that this means a well–known decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. From the decomposition we have

$$\mathrm{e}^{-itgA_2} = T \begin{pmatrix} 1 & \\ & \mathrm{e}^{-itgB_1} \end{pmatrix} T^{\dagger}. \tag{16}$$

Therefore to calculate e^{-itgA_2} we have only to do e^{-itgB_1}. Noting the relation

$$B_1^2 = \begin{pmatrix} 2(N+1) & 0 & 2a^2 \\ 0 & 2(2N+1) & 0 \\ 2(a^{\dagger})^2 & 0 & 2N \end{pmatrix},$$

$$B_1^3 = \begin{pmatrix} 2(2N+3) & & \\ & 2(2N+1) & \\ & & 2(2N-1) \end{pmatrix} B_1 \equiv DB_1, \tag{17}$$

and so

$$B_1^{2n} = D^{n-1} B_1^2 \quad \text{for} \quad n \geq 1, \quad B_1^{2n+1} = D^n B_1 \quad \text{for} \quad n \geq 0$$

we obtain by making use of the Taylor expansion

$$
\begin{aligned}
\mathrm{e}^{-itgB_1} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (tg)^{2n} B_1^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (tg)^{2n+1} B_1^{2n+1} \\
&= \begin{pmatrix} 1 + \frac{2N+2}{2N+3} f(N+1) & -ih(N+1)a & \frac{2}{2N+3} f(N+1)a^2 \\ -ih(N)a^\dagger & 1 + 2f(N) & -ih(N)a \\ \frac{2}{2N-1} f(N-1)(a^\dagger)^2 & -ih(N-1)a^\dagger & 1 + \frac{2N}{2N-1} f(N-1) \end{pmatrix}
\end{aligned}
\tag{18}
$$

where

$$
f(N) = \frac{-1 + \cos\left(tg\sqrt{2(2N+1)}\right)}{2}, \quad h(N) = \frac{\sin\left(tg\sqrt{2(2N+1)}\right)}{\sqrt{2N+1}}.
$$

Three Atoms Case In this case A in (10) is written as

$$
A_3 = \begin{pmatrix}
0 & a & a & 0 & a & 0 & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & a & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & 0 & a & 0 \\
0 & a^\dagger & a^\dagger & 0 & 0 & 0 & 0 & a \\
a^\dagger & 0 & 0 & 0 & 0 & a & a & 0 \\
0 & a^\dagger & 0 & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & a^\dagger & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & 0 & a^\dagger & 0 & a^\dagger & a^\dagger & 0
\end{pmatrix}. \tag{19}
$$

We would like to look for the explicit form of solution like (13), (18). If we set

$$
T = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{20}
$$

then it is not difficult to see

$$
T^\dagger A_3 T = \begin{pmatrix} 0 & a & & & & & & \\ a^\dagger & 0 & & & & & & \\ & & 0 & a & & & & \\ & & a^\dagger & 0 & & & & \\ & & & & 0 & \sqrt{3}a & 0 & 0 \\ & & & & \sqrt{3}a^\dagger & 0 & 2a & 0 \\ & & & & 0 & 2a^\dagger & 0 & \sqrt{3}a \\ & & & & 0 & 0 & \sqrt{3}a^\dagger & 0 \end{pmatrix} \equiv \begin{pmatrix} B_{1/2} & & \\ & B_{1/2} & \\ & & B_{3/2} \end{pmatrix}.
$$

This means a decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$. From the decomposition we have

$$
\mathrm{e}^{-itgA_3} = T \begin{pmatrix} \mathrm{e}^{-itgB_{1/2}} & & \\ & \mathrm{e}^{-itgB_{1/2}} & \\ & & \mathrm{e}^{-itgB_{3/2}} \end{pmatrix} T^\dagger. \tag{21}
$$

Therefore we have only to calculate $\mathrm{e}^{-itgB_{3/2}}$, which is however not easy. In this case there is no simple relation like (12) or (17), so we must find another one.

Let us state **the key lemma** for that. Noting

$$
B_{3/2}^2 = \begin{pmatrix} 3N+3 & 0 & 2\sqrt{3}a^2 & 0 \\ 0 & 7N+4 & 0 & 2\sqrt{3}a^2 \\ 2\sqrt{3}(a^\dagger)^2 & 0 & 7N+3 & 0 \\ 0 & 2\sqrt{3}(a^\dagger)^2 & 0 & 3N \end{pmatrix},
$$

$$
B_{3/2}^3 = \begin{pmatrix} 0 & \sqrt{3}(7N+11)a & 0 & 6a^3 \\ \sqrt{3}(7N+4)a^\dagger & 0 & 20(N+1)a & 0 \\ 0 & 20Na^\dagger & 0 & \sqrt{3}(7N+3)a \\ 6(a^\dagger)^3 & 0 & \sqrt{3}(7N-4)a^\dagger & 0 \end{pmatrix},
$$

and the relations

$$
B_{3/2}^{2n+1} = B_{3/2} B_{3/2}^{2n}, \quad B_{3/2}^{2n+2} = B_{3/2}^2 B_{3/2}^{2n},
$$

we can obtain $B_{3/2}^{2n}$ and $B_{3/2}^{2n+1}$ like

$$B_{3/2}^{2n} = \begin{pmatrix} \alpha_n(N+2) & 0 & 2\sqrt{3}\xi_n(N+2)a^2 & 0 \\ 0 & \beta_n(N+1) & 0 & 2\sqrt{3}\xi_n(N+1)a^2 \\ 2\sqrt{3}\xi_n(N)(a^\dagger)^2 & 0 & \gamma_n(N) & 0 \\ 0 & 2\sqrt{3}\xi_n(N-1)(a^\dagger)^2 & 0 & \delta_n(N-1) \end{pmatrix}, \tag{22}$$

$$B_{3/2}^{2n+1} = \begin{pmatrix} 0 & \sqrt{3}\beta_n(N+2)a & 0 & 6\xi_n(N+2)a^3 \\ \sqrt{3}\beta_n(N+1)a^\dagger & 0 & 2\xi_{n+1}(N+1)a & 0 \\ 0 & 2\xi_{n+1}(N)a^\dagger & 0 & \sqrt{3}\gamma_n(N)a \\ 6\xi_n(N-1)(a^\dagger)^3 & 0 & \sqrt{3}\gamma_n(N-1)a^\dagger & 0 \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} &\alpha_n(N) = (v_+\lambda_+^n - v_-\lambda_-^n)/(2\sqrt{d}), \quad \beta_n(N) = (w_+\lambda_+^n - w_-\lambda_-^n)/(2\sqrt{d}), \\ &\gamma_n(N) = (v_+\lambda_-^n - v_-\lambda_+^n)/(2\sqrt{d}), \quad \delta_n(N) = (w_+\lambda_-^n - w_-\lambda_+^n)/(2\sqrt{d}), \\ &\xi_n(N) = (\lambda_+^n - \lambda_-^n)/(2\sqrt{d}), \end{aligned}$$

and $\lambda_\pm \equiv \lambda_\pm(N)$, $v_\pm \equiv v_\pm(N)$, $w_\pm \equiv w_\pm(N)$, $d \equiv d(N)$ defined by

$$\begin{aligned} &\lambda_\pm(N) = 5N \pm \sqrt{d(N)},\ v_\pm(N) = -2N - 3 \pm \sqrt{d(N)},\ w_\pm(N) = 2N - 3 \pm \sqrt{d(N)}, \\ &d(N) = 16N^2 + 9. \end{aligned}$$

Then by making use of (22) and (23) we have

$$\begin{aligned} \mathrm{e}^{-itgB_{3/2}} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (tg)^{2n} B_{3/2}^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (tg)^{2n+1} B_{3/2}^{2n+1} \\ &= \begin{pmatrix} f_2(N+2) & -\sqrt{3}iF_1(N+2)a & 2\sqrt{3}h_1(N+2)a^2 & -6iH_0(N+2)a^3 \\ -\sqrt{3}iF_1(N+1)a^\dagger & f_1(N+1) & -2iH_1(N+1)a & 2\sqrt{3}h_1(N+1)a^2 \\ 2\sqrt{3}h_1(N)(a^\dagger)^2 & -2iH_1(N)a^\dagger & f_0(N) & -\sqrt{3}iF_0(N)a \\ -6iH_0(N-1)(a^\dagger)^3 & 2\sqrt{3}h_1(N-1)(a^\dagger)^2 & -\sqrt{3}iF_0(N-1)a^\dagger & f_{-1}(N-1) \end{pmatrix} \end{aligned} \tag{24}$$

where

$$
\begin{aligned}
f_2(N) &= \left\{ v_+(N)\mathrm{cos}(tg\sqrt{\lambda_+(N)}) - v_-(N)\mathrm{cos}(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}), \\
f_1(N) &= \left\{ w_+(N)\mathrm{cos}(tg\sqrt{\lambda_+(N)}) - w_-(N)\mathrm{cos}(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}), \\
f_0(N) &= \left\{ v_+(N)\mathrm{cos}(tg\sqrt{\lambda_-(N)}) - v_-(N)\mathrm{cos}(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}), \\
f_{-1}(N) &= \left\{ w_+(N)\mathrm{cos}(tg\sqrt{\lambda_-(N)}) - w_-(N)\mathrm{cos}(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}), \\
h_1(N) &= \left\{ \mathrm{cos}(tg\sqrt{\lambda_+(N)}) - \mathrm{cos}(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}), \\
F_1(N) &= \left\{ \frac{w_+(N)}{\sqrt{\lambda_+(N)}}\mathrm{sin}(tg\sqrt{\lambda_+(N)}) - \frac{w_-(N)}{\sqrt{\lambda_-(N)}}\mathrm{sin}(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}), \\
F_0(N) &= \left\{ \frac{v_+(N)}{\sqrt{\lambda_-(N)}}\mathrm{sin}(tg\sqrt{\lambda_-(N)}) - \frac{v_-(N)}{\sqrt{\lambda_+(N)}}\mathrm{sin}(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}), \\
H_1(N) &= \left\{ \sqrt{\lambda_+(N)}\mathrm{sin}(tg\sqrt{\lambda_+(N)}) - \sqrt{\lambda_-(N)}\mathrm{sin}(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}), \\
H_0(N) &= \left\{ \frac{1}{\sqrt{\lambda_+(N)}}\mathrm{sin}(tg\sqrt{\lambda_+(N)}) - \frac{1}{\sqrt{\lambda_-(N)}}\mathrm{sin}(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}).
\end{aligned}
$$

Now we must calculate the term

$$F(t) \equiv \mathrm{e}^{itgA_n} \left(\mathrm{e}^{it\omega S_3} \otimes \mathrm{e}^{it\omega N} \right) V(t) \left(\mathrm{e}^{-it\omega S_3} \otimes \mathrm{e}^{-it\omega N} \right) \mathrm{e}^{-itgA_n} \tag{25}$$

from (9), so we introduce a brief notation

$$\tilde{V}(t) \equiv \left(\mathrm{e}^{it\omega S_3} \otimes \mathrm{e}^{it\omega N} \right) V(t) \left(\mathrm{e}^{-it\omega S_3} \otimes \mathrm{e}^{-it\omega N} \right).$$

Therefore we want to calculate

$$F(t) = \mathrm{e}^{itgA_n}\tilde{V}(t)\mathrm{e}^{-itgA_n} = T(\text{block diagonals})T^\dagger\tilde{V}(t)T(\text{block diagonals})^\dagger T^\dagger$$

explicitly for the case of $n = 2$ and 3. For that let us calculate $\tilde{V}(t)$ and $T^\dagger\tilde{V}(t)T$ in advance. The calculation is straightforward and the result is

Two Atoms Case

$$\tilde{V}(t) = \begin{pmatrix} 0 & q(t) & p(t) & 0 \\ \bar{q}(t) & 0 & 0 & p(t) \\ \bar{p}(t) & 0 & 0 & q(t) \\ 0 & \bar{p}(t) & \bar{q}(t) & 0 \end{pmatrix} \otimes \mathbf{1} \tag{26}$$

$$\text{with} \quad p(t) \equiv h_1 \mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}, \quad q(t) \equiv h_2 \mathrm{e}^{i\{(\Omega_2+\omega)t+\phi_2\}}$$

and

$$T^\dagger \tilde{V}(t) T = \begin{pmatrix} 0 & \frac{-\bar{p}+\bar{q}}{\sqrt{2}} & 0 & \frac{p-q}{\sqrt{2}} \\ \frac{-p+q}{\sqrt{2}} & 0 & \frac{p+q}{\sqrt{2}} & 0 \\ 0 & \frac{\bar{p}+\bar{q}}{\sqrt{2}} & 0 & \frac{p+q}{\sqrt{2}} \\ \frac{\bar{p}-\bar{q}}{\sqrt{2}} & 0 & \frac{\bar{p}+\bar{q}}{\sqrt{2}} & 0 \end{pmatrix} \otimes \mathbf{1} \tag{27}$$

where we have omitted the time t for simplicity.

Three Atoms Case

$$\tilde{V}(t) = \begin{pmatrix} 0 & r(t) & q(t) & 0 & p(t) & 0 & 0 & 0 \\ \bar{r}(t) & 0 & 0 & q(t) & 0 & p(t) & 0 & 0 \\ \bar{q}(t) & 0 & 0 & r(t) & 0 & 0 & p(t) & 0 \\ 0 & \bar{q}(t) & \bar{r}(t) & 0 & 0 & 0 & 0 & p(t) \\ \bar{p}(t) & 0 & 0 & 0 & 0 & r(t) & q(t) & 0 \\ 0 & \bar{p}(t) & 0 & 0 & \bar{r}(t) & 0 & 0 & q(t) \\ 0 & 0 & \bar{p}(t) & 0 & \bar{q}(t) & 0 & 0 & r(t) \\ 0 & 0 & 0 & \bar{p}(t) & 0 & \bar{q}(t) & \bar{r}(t) & 0 \end{pmatrix} \otimes \mathbf{1} \tag{28}$$

$$\text{with} \quad p(t) \equiv h_1 \mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}, \quad q(t) \equiv h_2 \mathrm{e}^{i\{(\Omega_2+\omega)t+\phi_2\}}, \quad r(t) \equiv h_3 \mathrm{e}^{i\{(\Omega_3+\omega)t+\phi_3\}}$$

and

$$T^\dagger \tilde{V}(t) T = \begin{pmatrix} 0 & p & 0 & \frac{q-r}{\sqrt{3}} & \frac{-\bar{q}+\bar{r}}{\sqrt{2}} & 0 & \frac{q-r}{\sqrt{6}} & 0 \\ \bar{p} & 0 & \frac{\bar{q}-\bar{r}}{\sqrt{3}} & 0 & 0 & \frac{-\bar{q}+\bar{r}}{\sqrt{6}} & 0 & \frac{q-r}{\sqrt{2}} \\ 0 & \frac{q-r}{\sqrt{3}} & 0 & \frac{-p+2q+2r}{3} & \frac{-2\bar{p}+\bar{q}+\bar{r}}{\sqrt{6}} & 0 & \frac{2p-q-r}{3\sqrt{2}} & 0 \\ \frac{\bar{q}-\bar{r}}{\sqrt{3}} & 0 & \frac{-\bar{p}+2\bar{q}+2\bar{r}}{3} & 0 & 0 & \frac{-2\bar{p}+\bar{q}+\bar{r}}{3\sqrt{2}} & 0 & \frac{2p-q-r}{\sqrt{6}} \\ \frac{-q+r}{\sqrt{2}} & 0 & \frac{-2p+q+r}{\sqrt{6}} & 0 & 0 & \frac{p+q+r}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-q+r}{\sqrt{6}} & 0 & \frac{-2p+q+r}{3\sqrt{2}} & \frac{\bar{p}+\bar{q}+\bar{r}}{\sqrt{3}} & 0 & \frac{2(p+q+r)}{3} & 0 \\ \frac{\bar{q}-\bar{r}}{\sqrt{6}} & 0 & \frac{2\bar{p}-\bar{q}-\bar{r}}{3\sqrt{2}} & 0 & 0 & \frac{2(\bar{p}+\bar{q}+\bar{r})}{3} & 0 & \frac{p+q+r}{\sqrt{3}} \\ 0 & \frac{\bar{q}-\bar{r}}{\sqrt{2}} & 0 & \frac{2\bar{p}-\bar{q}-\bar{r}}{\sqrt{6}} & 0 & 0 & \frac{\bar{p}+\bar{q}+\bar{r}}{\sqrt{3}} & 0 \end{pmatrix} \otimes \mathbf{1} \tag{29}$$

where we have omitted the time t for simplicity.

To write down all components of $F(t) = \mathrm{e}^{itgA_n} \tilde{V}(t) \mathrm{e}^{-itgA_n}$ is very long and we moreover don't need all of them, so we omit it here. Next let us go to our quantum computation based on a few atoms of laser–cooled and trapped linearly in a cavity.

3 Quantum Computation

To develop a quantum computation based on atoms laser–cooled and trapped **linearly** in a cavity (Cavity QED Quantum Computation) a quick and clear construction of the both controlled NOT gate and controlled–controlled NOT gate is required, [5]. Let us construct such very important quantum gates in this section.

3.1 Controlled NOT Gate

In this subsection we treat the case of two atoms (the system of two qubits). First let us make a short review of the system of two–qubits. Each element can be written as

$$\psi = a_{++}|+\rangle \otimes |+\rangle + a_{+-}|+\rangle \otimes |-\rangle + a_{-+}|-\rangle \otimes |+\rangle + a_{--}|-\rangle \otimes |-\rangle$$

with two bases $|+\rangle$ and $|-\rangle$ and $|a_{++}|^2+|a_{+-}|^2+|a_{-+}|^2+|a_{--}|^2=1$. Here if we identify

$$|+\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |-\rangle=\begin{pmatrix}0\\1\end{pmatrix},$$

then ψ above becomes

$$\psi=\begin{pmatrix}a_{++}\\a_{+-}\\a_{-+}\\a_{--}\end{pmatrix}. \tag{30}$$

How do we embed two–qubits in our quantized system ? It is not known at the current time, which may depend on methods of experimentalists. Therefore let us consider the simplest one like

$$|\psi(t)\rangle=\begin{pmatrix}a_{++}(t)\\a_{+-}(t)\\a_{-+}(t)\\a_{--}(t)\end{pmatrix}\otimes|0\rangle, \tag{31}$$

where $|0\rangle$ is the ground state of the radiation field $(a|0\rangle=0)$. We note that in full theory we must consider the following superpositions

$$|\Psi(t)\rangle=\sum_{n=0}^{\infty}\begin{pmatrix}a_{++,n}(t)\\a_{+-,n}(t)\\a_{-+,n}(t)\\a_{--,n}(t)\end{pmatrix}\otimes|n\rangle$$

as a wave function, which is however too complicated to solve.

To determine a dynamics that the coefficients $a_{++},a_{+-},a_{-+},a_{--}$ will satisfy we substitute (31) into the equation

$$\begin{aligned}i\frac{d}{dt}|\psi(t)\rangle&=F(t)|\psi(t)\rangle=\mathrm{e}^{itgA_2}\tilde{V}(t)\mathrm{e}^{-itgA_2}|\psi(t)\rangle\\&=T\begin{pmatrix}1&\\&\mathrm{e}^{itgB_1}\end{pmatrix}T^{\dagger}\tilde{V}(t)T\begin{pmatrix}1&\\&\mathrm{e}^{-itgB_1}\end{pmatrix}T^{\dagger}|\psi(t)\rangle.\end{aligned}$$

Let us rewrite the above equation. If we set

$$|\varphi(t)\rangle \equiv T^\dagger|\psi(t)\rangle \Longleftrightarrow \begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix} \otimes |0\rangle \equiv \begin{pmatrix} \frac{a_{+-}(t)-a_{-+}(t)}{\sqrt{2}} \\ a_{++}(t) \\ \frac{a_{+-}(t)+a_{-+}(t)}{\sqrt{2}} \\ a_{--}(t) \end{pmatrix} \otimes |0\rangle \tag{32}$$

then

$$i\frac{d}{dt}|\varphi(t)\rangle = \begin{pmatrix} 1 & \\ & \mathrm{e}^{itgB_1} \end{pmatrix} T^\dagger \tilde{V}(t) T \begin{pmatrix} 1 & \\ & \mathrm{e}^{-itgB_1} \end{pmatrix} |\varphi(t)\rangle. \tag{33}$$

On the other hand, we have calculated the term $T^\dagger \tilde{V}(t) T$ in (27).

Note that the above equation is not satisfied under the restrictive ansatz (32). Because the left hand side contains only the ground state $|0\rangle$, while the right hand side contains the ground state $|0\rangle$ and some excited states $|1\rangle$, $|2\rangle$, $|3\rangle$. However, the states $|1\rangle$, $|2\rangle$, $|3\rangle$ which have no corresponding kinetic terms contain the coupling constants h_1 and h_2 (see $p(t)$ and $q(t)$ in (26)), so the equation is approximately satisfied if they are **small enough** (namely, in the weak coupling regime in the title).

Therefore the (full) equation is reduced to the equations of $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ at the ground state $|0\rangle$:

$$i\frac{d}{dt}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix} = \begin{pmatrix} 0 & x_{12} & 0 & x_{14} \\ x_{21} & 0 & x_{23} & 0 \\ 0 & x_{32} & 0 & x_{34} \\ x_{41} & 0 & x_{43} & 0 \end{pmatrix} \begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix} \tag{34}$$

where

$$\begin{aligned} x_{12} &= \bar{x}_{21}, \\ x_{14} &= \frac{p-q}{\sqrt{2}}, \\ x_{21} &= \frac{-p+q}{\sqrt{2}}\left(1+\frac{2}{3}f(1)\right), \\ x_{23} &= \frac{p+q}{\sqrt{2}}\left(1+2f(0)+\frac{2}{3}f(1)+\frac{4}{3}f(0)f(1)+h(0)h(1)\right), \\ x_{32} &= \bar{x}_{23}, \end{aligned}$$

$$
\begin{aligned}
x_{34} &= \frac{p+q}{\sqrt{2}}\left(1+2f(0)\right), \\
x_{41} &= \bar{x}_{14}, \\
x_{43} &= \bar{x}_{34}
\end{aligned}
$$

and

$$
\begin{aligned}
&p = h_1 \mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}, \quad q = h_2 \mathrm{e}^{i\{(\Omega_2+\omega)t+\phi_2\}} \\
&f(0) = \frac{-1+\cos(\sqrt{2}gt)}{2}, \quad f(1) = \frac{-1+\cos(\sqrt{6}gt)}{2}, \\
&h(0) = \sin(\sqrt{2}gt), \quad h(1) = \frac{\sin(\sqrt{6}gt)}{\sqrt{3}}.
\end{aligned}
$$

How do we solve it ? We use some resonance condition and the rotating wave approximation associated to it, which is popular in quantum optics or in a field of laser physics. Let us focus on the (2,3)–component of the matrix which came from the interaction term. The products $f(0)f(1)$ and $h(0)h(1)$ contain the term $\mathrm{e}^{-itg(\sqrt{2}+\sqrt{6})}$ by the Euler formulas $\cos(\theta) = (\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta})/2,\ \sin(\theta) = (\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta})/2i.$ Noting

$$
\mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}\mathrm{e}^{-itg(\sqrt{2}+\sqrt{6})} = \mathrm{e}^{i\{(\Omega_1+\omega-(\sqrt{2}+\sqrt{6})g)t+\phi_1\}},
$$

we set a new resonance condition

$$
\Omega_1 + \omega - (\sqrt{2}+\sqrt{6})g = 0. \tag{35}
$$

All terms in (34) except for the constant one $\mathrm{e}^{i\{(\Omega_1+\omega-(\sqrt{2}+\sqrt{6})g)t+\phi_1\}} = \mathrm{e}^{i\phi_1}$ contain ones like $\mathrm{e}^{i(t\theta+\alpha)}$ $(\theta \neq 0)$, so we neglect all such oscillating terms (a rotating wave approximation). Then (34) reduces to a very simple matrix equation

$$
i\frac{d}{dt}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix} = \frac{-\sqrt{2}(\sqrt{3}-1)h_1}{24}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{i\phi_1} & 0 \\ 0 & \mathrm{e}^{-i\phi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix}. \tag{36}
$$

The solution is easily obtained to be

$$\begin{aligned}
\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix}
&= \exp\left\{ \frac{i(\sqrt{6}-\sqrt{2})h_1 t}{24} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{i\phi_1} & 0 \\ 0 & \mathrm{e}^{-i\phi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & i\mathrm{e}^{i\phi_1}\sin(\alpha t) & 0 \\ 0 & i\mathrm{e}^{-i\phi_1}\sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} \\
&\equiv U(t) \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix}
\end{aligned} \tag{37}$$

where we have set $\alpha = \frac{\sqrt{6}-\sqrt{2}}{24}h_1$. That is, we obtained the unitary operator $U(t)$. In particular, if we choose t_0 and ϕ_1 satisfying

$$\cos(\alpha t_0) = 0 \ (\sin(\alpha t_0) = 1) \quad \text{and} \quad \mathrm{e}^{i\phi_1} = i$$

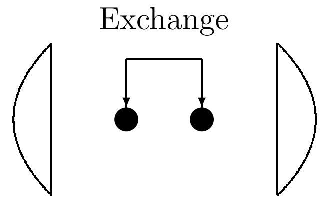
then

$$U(t_0) = \begin{pmatrix} 1 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}. \tag{38}$$

From this we want to construct the controlled NOT operator. However, it is not easy [2].

At this stage we use a very skillful method due to Dirac [10] . That is, we exchange two atoms in the cavity

[2] $U(t_0)$ is imprimitive in the sense of [9], so the main theorem in it says that our system is universal (namely, we can construct any element in $U(4)$). However, how to construct a unitary element explicitly is not given in [9]



which introduces the exchange (swap) operator

$$P = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}. \tag{39}$$

Multiplying $U(t_0)$ by P gives

$$PU(t_0) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

and from this we obtain

$$(\mathbf{1}_2 \otimes \sigma_1)PU(t_0)(\mathbf{1}_2 \otimes \sigma_1) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{40}$$

This is just the controlled σ_z operator. From this it is easy to construct the controlled NOT operator, namely

$$C_{NOT} = (\mathbf{1}_2 \otimes W)C_{\sigma_z}(\mathbf{1}_2 \otimes W) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

where W is the Walsh–Hadamard operator given by

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1}. \tag{41}$$

See for example [6]. As to a construction of W by making use of Rabi oscillations see Appendix or [11].

Therefore our system is universal [5], [9].

A comment is in order.

(a) If we choose t_0 in $U(t)$ satisfying

$$\cos(\alpha t_0) = -1 \quad (\sin(\alpha t_0) = 0)$$

then we obtain the matrix

$$U(t_0) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \sigma_3 \otimes \sigma_3.$$

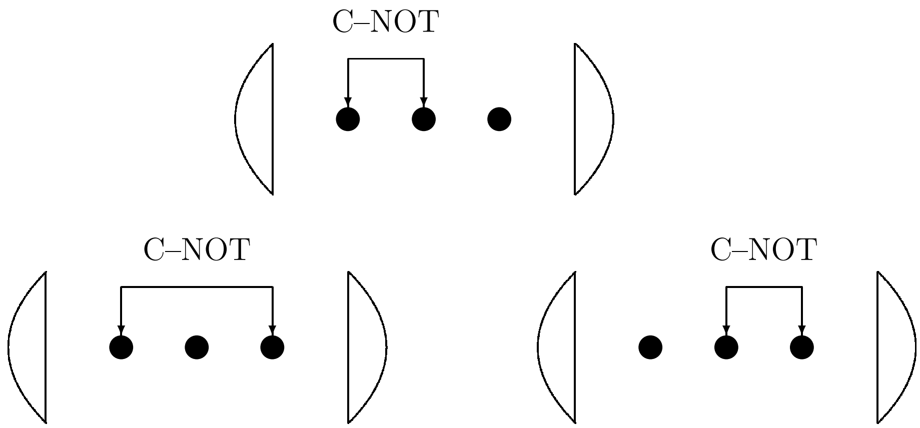
(b) In place of the ansatz (31) we can set for example

$$|\psi(t)\rangle = \begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ 0 \\ 0 \end{pmatrix} \otimes |0\rangle + \begin{pmatrix} 0 \\ 0 \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} \otimes |1\rangle.$$

Then we can trace the same line shown in this section and obtain a unitary operator under some resonance condition like (35). This is a good exercise, so we leave it to the readers.

3.2 Controlled-Controlled NOT Gate

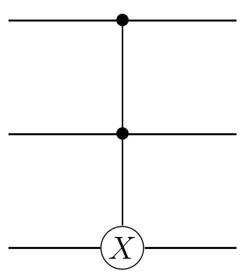
In this subsection we treat the case of three atoms (the system of three qubits). To perform a quantum computation we need a (rapid) construction of the the controlled–controlled NOT gate. To construct it we need a construction of the controlled NOT gates of three types like



The controlled–controlled NOT gate (operator)

$$CC_{NOT} \ : \ \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \longrightarrow \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$$

is shown as a picture

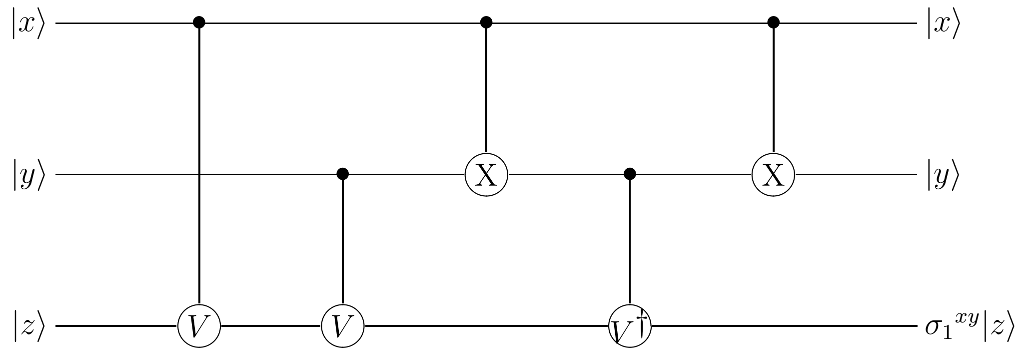


$= \quad CC_{NOT}$

or in a matrix form

$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}.$$

The (usual) construction by making use of controlled NOT or controlled U gates is shown as a picture ([6], [5])



where V is a unitary matrix given by

$$V = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \quad \Longrightarrow \quad V^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 .$$

However, we have not seen "realistic" constructions in any references, so we give the explicit construction. See the figure 2.

To embed three–qubits in our quantized system we consider the simplest one as a wave function

$$|\psi(t)\rangle = \begin{pmatrix} a_{+++}(t) \\ a_{++-}(t) \\ a_{+-+}(t) \\ a_{+--}(t) \\ a_{-++}(t) \\ a_{-+-}(t) \\ a_{--+}(t) \\ a_{---}(t) \end{pmatrix} \otimes |0\rangle . \qquad (42)$$

like in the case of two–qubits (31).

To determine a dynamics that the coefficients $a_{+++}, a_{++-}, \cdots, a_{---}$ satisfy we substitute (42) into the equation

$$i\frac{d}{dt}|\psi(t)\rangle = F(t)|\psi(t)\rangle = \mathrm{e}^{itgA_3}\tilde{V}(t)\mathrm{e}^{-itgA_3}|\psi(t)\rangle$$

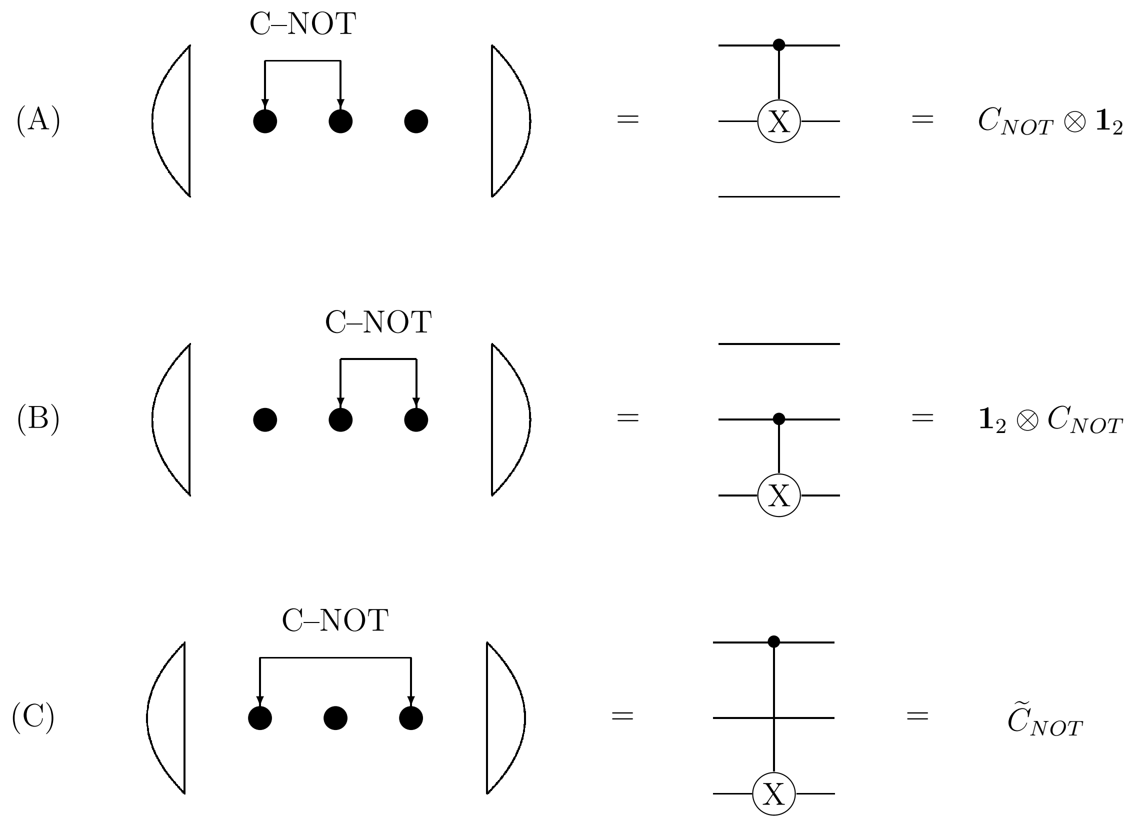


Figure 2: The Controlled NOT gates of three types ((A), (B), (C) from the above) for the three atoms in the cavity

$$= T \begin{pmatrix} \mathrm{e}^{itgB_{1/2}} & & \\ & \mathrm{e}^{itgB_{1/2}} & \\ & & \mathrm{e}^{itgB_{3/2}} \end{pmatrix} T^{\dagger}\tilde{V}(t)T \begin{pmatrix} \mathrm{e}^{-itgB_{1/2}} & & \\ & \mathrm{e}^{-itgB_{1/2}} & \\ & & \mathrm{e}^{-itgB_{3/2}} \end{pmatrix} T^{\dagger}|\psi(t)\rangle.$$

Let us rewrite the above equation. If we set

$$|\varphi(t)\rangle \equiv T^{\dagger}|\psi(t)\rangle \Longleftrightarrow \begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix} \otimes |0\rangle \equiv \begin{pmatrix} \frac{a_{++-}(t)-a_{+-+}(t)}{\sqrt{2}} \\ \frac{a_{-+-}(t)-a_{--+}(t)}{\sqrt{2}} \\ \frac{a_{++-}(t)+a_{+-+}(t)-2a_{-++}}{\sqrt{6}} \\ \frac{2a_{+--}(t)-a_{-+-}(t)-a_{--+}}{\sqrt{6}} \\ a_{+++} \\ \frac{a_{++-}(t)+a_{+-+}(t)+a_{-++}}{\sqrt{3}} \\ \frac{a_{+--}(t)+a_{-+-}(t)+a_{--+}}{\sqrt{3}} \\ a_{---}(t) \end{pmatrix} \otimes |0\rangle \qquad (43)$$

then we have

$$i\frac{d}{dt}|\varphi(t)\rangle = \begin{pmatrix} \mathrm{e}^{itgB_{1/2}} & & \\ & \mathrm{e}^{itgB_{1/2}} & \\ & & \mathrm{e}^{itgB_{3/2}} \end{pmatrix} T^{\dagger}\tilde{V}(t)T \begin{pmatrix} \mathrm{e}^{-itgB_{1/2}} & & \\ & \mathrm{e}^{-itgB_{1/2}} & \\ & & \mathrm{e}^{-itgB_{3/2}} \end{pmatrix} |\varphi(t)\rangle. \qquad (44)$$

On the other hand, we have calculated $\mathrm{e}^{-itgB_{1/2}}$ in (13), $\mathrm{e}^{-itgB_{3/2}}$ in (24) and the middle term $T^{\dagger}\tilde{V}(t)T$ in (29).

Therefore the (full) equation is reduced to the equations of $\{\varphi_0, \varphi_1, \cdots, \varphi_7\}$ at the ground state $|0\rangle$:

$$i\frac{d}{dt}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix} = \begin{pmatrix} 0 & x_{12} & 0 & x_{14} & x_{15} & 0 & x_{17} & 0 \\ x_{21} & 0 & x_{23} & 0 & 0 & x_{26} & 0 & x_{28} \\ 0 & x_{32} & 0 & x_{34} & x_{35} & 0 & x_{37} & 0 \\ x_{41} & 0 & x_{43} & 0 & 0 & x_{46} & 0 & x_{48} \\ x_{51} & 0 & x_{53} & 0 & 0 & x_{56} & 0 & 0 \\ 0 & x_{62} & 0 & x_{64} & x_{65} & 0 & x_{67} & 0 \\ x_{71} & 0 & x_{73} & 0 & 0 & x_{76} & 0 & x_{78} \\ 0 & x_{82} & 0 & x_{84} & 0 & 0 & x_{87} & 0 \end{pmatrix} \begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix} \tag{45}$$

where

$$\begin{aligned}
x_{12} &= pC(0)C(1), \\
x_{14} &= \frac{q-r}{\sqrt{3}}C(0)C(1), \\
x_{15} &= \bar{x}_{51}, \\
x_{17} &= \frac{q-r}{\sqrt{6}}\left(f_0(0)C(1) + 3F_0(0)S(1)\right), \\
x_{21} &= \bar{x}_{12}, \\
x_{23} &= \bar{x}_{32}, \\
x_{26} &= \bar{x}_{62}, \\
x_{28} &= \frac{q-r}{\sqrt{2}}f_{-1}(-1)C(0), \\
x_{32} &= \frac{q-r}{\sqrt{3}}C(0)C(1), \\
x_{34} &= \frac{-p+2q+2r}{3}C(0)C(1), \\
x_{35} &= \bar{x}_{53}, \\
x_{37} &= \frac{2p-q-r}{3\sqrt{2}}\left(f_0(0)C(1) + 3F_0(0)S(1)\right), \\
x_{41} &= \bar{x}_{14}, \\
x_{43} &= \bar{x}_{34}, \\
x_{46} &= \bar{x}_{64},
\end{aligned}$$

$$
\begin{aligned}
x_{48} &= \frac{2p-q-r}{\sqrt{6}} f_{-1}(-1)C(0),\\
x_{51} &= \frac{-q+r}{\sqrt{2}} \left(C(1)f_2(2)+S(1)F_1(2)\right),\\
x_{53} &= \frac{-2p+q+r}{\sqrt{6}} \left(C(1)f_2(2)+S(1)F_1(2)\right),\\
x_{56} &= \frac{p+q+r}{\sqrt{3}} \left(f_1(1)f_2(2)+4H_1(1)F_1(2)+24h_1(1)h_1(2)\right),\\
x_{62} &= \frac{-q+r}{\sqrt{2}} C(0)f_1(1),\\
x_{64} &= \frac{-2p+q+r}{3\sqrt{2}} C(0)f_1(1),\\
x_{65} &= \bar{x}_{56},\\
x_{67} &= \frac{2(p+q+r)}{3} \left(f_0(0)f_1(1)+3F_0(0)H_1(1)\right),\\
x_{71} &= \bar{x}_{17},\\
x_{73} &= \bar{x}_{37},\\
x_{76} &= \bar{x}_{67},\\
x_{78} &= \frac{p+q+r}{\sqrt{3}} f_{-1}(-1)f_0(0),\\
x_{82} &= \bar{x}_{28},\\
x_{84} &= \bar{x}_{48},\\
x_{87} &= \bar{x}_{78}
\end{aligned}
$$

and $C(0)$, $C(1)$, $S(1)$, $f_{-1}(-1)$, $f_0(0)$, $f_1(1)$, $f_2(2)$, $F_0(0)$, $F_1(2)$, $h_1(1)$, $h_1(2)$, $H_1(1)$ are respectively given as

$$
\begin{aligned}
p(t) &= h_1 \mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}, \quad q(t) = h_2 \mathrm{e}^{i\{(\Omega_2+\omega)t+\phi_2\}}, \quad r(t) = h_3 \mathrm{e}^{i\{(\Omega_3+\omega)t+\phi_3\}},\\
C(0) &= 1,\\
C(1) &= \cos(tg),\\
S(1) &= \sin(tg),\\
f_{-1}(-1) &= 1,\\
f_0(0) &= \cos(tg\sqrt{3}),\\
f_1(1) &= \frac{2\cos(tg\sqrt{10})+3}{5},
\end{aligned}
$$

$$
\begin{aligned}
f_2(2) &= \frac{(-7+\sqrt{73})\cos(tg\sqrt{10+\sqrt{73}}) + (7+\sqrt{73})\cos(tg\sqrt{10-\sqrt{73}})}{2\sqrt{73}}, \\
F_0(0) &= \frac{\sin(tg\sqrt{3})}{\sqrt{3}}, \\
F_1(2) &= \frac{1}{2\sqrt{73}}\left\{\frac{1+\sqrt{73}}{\sqrt{10+\sqrt{73}}}\sin(tg\sqrt{10+\sqrt{73}}) - \frac{1-\sqrt{73}}{\sqrt{10-\sqrt{73}}}\sin(tg\sqrt{10-\sqrt{73}})\right\}, \\
f_3(1) &= \frac{\cos(tg\sqrt{10})-1}{10}, \\
f_3(2) &= \frac{\cos(tg\sqrt{10+\sqrt{73}}) - \cos(tg\sqrt{10-\sqrt{73}})}{2\sqrt{73}}, \\
H_1(1) &= \frac{\sin(tg\sqrt{10})}{\sqrt{10}}.
\end{aligned}
$$

These equations are complicated enough.

Next let us derive the three controlled NOT gates (A), (B), (C) in Figure 2 from the matrix equation above. For that we use a resonance condition and rotating wave approximation associated to it.

Derivation of (A)

We focus on the components x_{56} and $x_{65} = \bar{x}_{56}$ in the matrix. $f_1(1)f_2(2)$ and $h_1(1)h_1(2)$ contain the term $\mathrm{e}^{-itg\sqrt{10+\sqrt{73}}}$ coming from the Euler formula, and therefore x_{56} contain the oscillating term [3]

$$
\mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}\mathrm{e}^{-itg\sqrt{10+\sqrt{73}}} = \mathrm{e}^{i\{(\Omega_1+\omega-g\sqrt{10+\sqrt{73}})t+\phi_1\}}.
$$

We note that this term is not contained in other components in the matrix. Here we set the resonance condition

$$
\Omega_1 + \omega - g\sqrt{10+\sqrt{73}} = 0 \tag{46}
$$

and apply the rotating wave approximation associated to this. Then the above complicated matrix equation becomes a very simple one

[3] We use $p(t)$ in the first atom, however it is of course possible to use $q(t)$ (the second atom) or $r(t)$ (the third atom)

$$
i\frac{d}{dt}\begin{pmatrix}\varphi_0(t)\\ \varphi_1(t)\\ \varphi_2(t)\\ \varphi_3(t)\\ \varphi_4(t)\\ \varphi_5(t)\\ \varphi_6(t)\\ \varphi_7(t)\end{pmatrix} = \frac{\sqrt{3}(-11+\sqrt{73})h_1}{20\sqrt{73}}\begin{pmatrix}0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\mathrm{e}^{i\phi_1}&0&0\\ 0&0&0&0&\mathrm{e}^{-i\phi_1}&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\end{pmatrix}\begin{pmatrix}\varphi_0(t)\\ \varphi_1(t)\\ \varphi_2(t)\\ \varphi_3(t)\\ \varphi_4(t)\\ \varphi_5(t)\\ \varphi_6(t)\\ \varphi_7(t)\end{pmatrix}. \quad (47)
$$

The solution is easily obtained to be

$$
\begin{pmatrix}\varphi_0(t)\\ \varphi_1(t)\\ \varphi_2(t)\\ \varphi_3(t)\\ \varphi_4(t)\\ \varphi_5(t)\\ \varphi_6(t)\\ \varphi_7(t)\end{pmatrix} = \exp\left\{it\frac{\sqrt{3}(11-\sqrt{73})h_1}{20\sqrt{73}}\begin{pmatrix}0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\mathrm{e}^{i\phi_1}&0&0\\ 0&0&0&0&\mathrm{e}^{-i\phi_1}&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\end{pmatrix}\right\}\begin{pmatrix}\varphi_0(0)\\ \varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\\ \varphi_4(0)\\ \varphi_5(0)\\ \varphi_6(0)\\ \varphi_7(0)\end{pmatrix}
$$

$$
= \begin{pmatrix}1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&\cos(\alpha t)&i\mathrm{e}^{i\phi_1}\sin(\alpha t)&0&0\\ 0&0&0&0&i\mathrm{e}^{-i\phi_1}\sin(\alpha t)&\cos(\alpha t)&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}\varphi_0(0)\\ \varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\\ \varphi_4(0)\\ \varphi_5(0)\\ \varphi_6(0)\\ \varphi_7(0)\end{pmatrix}
$$

$$
\equiv U_A(t) \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \\ \varphi_5(0) \\ \varphi_6(0) \\ \varphi_7(0) \end{pmatrix} \tag{48}
$$

where we have set $\alpha = \frac{\sqrt{3}(11-\sqrt{73})h_1}{20\sqrt{73}}$.

Here if we choose t_A as $\cos(\alpha t_A) = -1$, then

$$
U_A(t_A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.
$$

By multiplying $\mathbf{1}_2 \otimes \sigma_1 \otimes \mathbf{1}_2$ from both sides we have

$$
\tilde{U}_A(t_A) \equiv (\mathbf{1}_2 \otimes \sigma_1 \otimes \mathbf{1}_2) U_A(t_A) (\mathbf{1}_2 \otimes \sigma_1 \otimes \mathbf{1}_2) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}. \tag{49}
$$

Moreover, by multiplying $\mathbf{1}_2 \otimes W \otimes \mathbf{1}_2$ from both sides we finally obtain the case (A) in Figure 2

$$(\mathbf{1}_2 \otimes W \otimes \mathbf{1}_2)\tilde{U}_A(t_A)(\mathbf{1}_2 \otimes W \otimes \mathbf{1}_2) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = C_{NOT} \otimes \mathbf{1}_2. \quad (50)$$

Derivation of (B)

We focus on the components x_{37} and $x_{73} = \bar{x}_{37}$ in the matrix. $f_0(0)C(1)$ and $F_0(0)S(1)$ contain the term $\mathrm{e}^{-itg(1+\sqrt{3})}$ coming from the Euler formula, and therefore x_{37} contain the oscillating term

$$\mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}\mathrm{e}^{-itg(1+\sqrt{3})} = \mathrm{e}^{i\{\left(\Omega_1+\omega-g(1+\sqrt{3})\right)t+\phi_1\}}.$$

This term is not contained in other components in the matrix. Here we set the resonance condition

$$\Omega_1 + \omega - g(1+\sqrt{3}) = 0 \quad (51)$$

and apply the rotating wave approximation associated to this. Then the above complicated matrix equation becomes a very simple one

$$i\frac{d}{dt}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix} = \frac{\sqrt{2}(1-\sqrt{3})h_1}{12}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{i\phi_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-i\phi_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix}. \quad (52)$$

The solution is easily obtained to be

$$\begin{pmatrix}\varphi_0(t)\\\varphi_1(t)\\\varphi_2(t)\\\varphi_3(t)\\\varphi_4(t)\\\varphi_5(t)\\\varphi_6(t)\\\varphi_7(t)\end{pmatrix} = \exp\left\{ it\frac{\sqrt{2}(-1+\sqrt{3})h_1}{12}\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\mathrm{e}^{i\phi_1}&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&\mathrm{e}^{-i\phi_1}&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\end{pmatrix}\right\}\begin{pmatrix}\varphi_0(0)\\\varphi_1(0)\\\varphi_2(0)\\\varphi_3(0)\\\varphi_4(0)\\\varphi_5(0)\\\varphi_6(0)\\\varphi_7(0)\end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&\cos(\beta t)&0&0&0&i\mathrm{e}^{i\phi_1}\sin(\beta t)&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&i\mathrm{e}^{-i\phi_1}\sin(\beta t)&0&0&0&\cos(\beta t)&0\\ 0&0&0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}\varphi_0(0)\\\varphi_1(0)\\\varphi_2(0)\\\varphi_3(0)\\\varphi_4(0)\\\varphi_5(0)\\\varphi_6(0)\\\varphi_7(0)\end{pmatrix}$$

$$\equiv U_B(t)\begin{pmatrix}\varphi_0(0)\\\varphi_1(0)\\\varphi_2(0)\\\varphi_3(0)\\\varphi_4(0)\\\varphi_5(0)\\\varphi_6(0)\\\varphi_7(0)\end{pmatrix} \tag{53}$$

where we have set $\beta = \frac{\sqrt{2}(-1+\sqrt{3})h_1}{12}$.

Here if we choose t_B as $\cos(\beta t_B) = -1$, then

$$U_B(t_B) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

By multiplying $\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_1$ from both sides we have

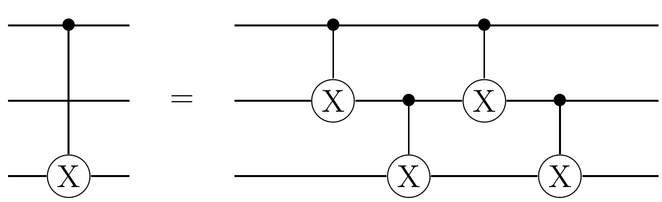
$$\tilde{U}_B(t_B) \equiv (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_1) U_B(t_B) (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_1) = \mathbf{1}_2 \otimes \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Moreover, by multiplying $\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes W$ from both sides we finally obtain the case (B) in Figure 2

$$(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes W) \tilde{U}_B(t_B) (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes W) = \mathbf{1}_2 \otimes \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = \mathbf{1}_2 \otimes C_{NOT}. \tag{54}$$

Derivation of (C)

It is well–known that the case (C) is obtained as



by making use of (A) and (B). However, 4 steps are needed in construction. We give a direct construction like (A) or (B) which is favorable from the point of view of quick construction.

We focus on the components x_{67} and $x_{76} = \bar{x}_{67}$ in the matrix. $f_0(0)f_1(1)$ and $F_0(0)H_1(1)$ contain the term $\mathrm{e}^{-itg(\sqrt{3}+\sqrt{10})}$ coming from the Euler formula, and therefore x_{67} contain the oscillating term

$$\mathrm{e}^{i\{(\Omega_1+\omega)t+\phi_1\}}\mathrm{e}^{-itg(\sqrt{3}+\sqrt{10})} = \mathrm{e}^{i\{\left(\Omega_1+\omega-g(\sqrt{3}+\sqrt{10})\right)t+\phi_1\}}.$$

This term is not contained in other components in the matrix. Here we set the resonance condition

$$\Omega_1 + \omega - g(\sqrt{3}+\sqrt{10}) = 0 \tag{55}$$

and apply the rotating wave approximation associated to this. Then the above complicated matrix equation becomes a very simple one

$$i\frac{d}{dt}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix} = \frac{(4-\sqrt{30})h_1}{60}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{i\phi_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-i\phi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix}. \tag{56}$$

The solution is easily obtained to be

$$
\begin{aligned}
\begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \\ \varphi_6(t) \\ \varphi_7(t) \end{pmatrix}
&= \exp\left\{ it\frac{(-4+\sqrt{30})h_1}{60} \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\mathrm{e}^{i\phi_1}&0 \\ 0&0&0&0&0&\mathrm{e}^{-i\phi_1}&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \right\} \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \\ \varphi_5(0) \\ \varphi_6(0) \\ \varphi_7(0) \end{pmatrix} \\
&= \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&\cos(\gamma t)&i\mathrm{e}^{i\phi_1}\sin(\gamma t)&0 \\ 0&0&0&0&0&i\mathrm{e}^{-i\phi_1}\sin(\gamma t)&\cos(\gamma t)&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \\ \varphi_5(0) \\ \varphi_6(0) \\ \varphi_7(0) \end{pmatrix} \\
&\equiv U_C(t) \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \\ \varphi_5(0) \\ \varphi_6(0) \\ \varphi_7(0) \end{pmatrix}
\end{aligned}
\tag{57}
$$

where we have set $\gamma = \frac{(-4+\sqrt{30})h_1}{60}$.

Here if we choose t_C as $\cos(\gamma t_C) = -1$, then

$$U_C(t_C) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By multiplying $\tilde{U}_A(t_A)$ in (49) from the left hand side we have

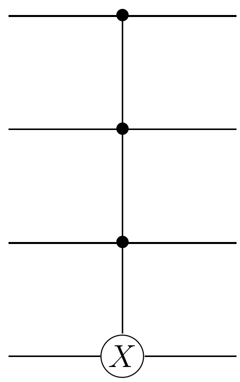
$$\tilde{U}_C(t_C) \equiv \tilde{U}_A(t_A)U_C(t_C) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Moreover, by multiplying $\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes W$ from both sides we finally obtain the case (C) in Figure 2

$$(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes W)\tilde{U}_C(t_C)(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes W) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \tilde{C}_{NOT}. \tag{58}$$

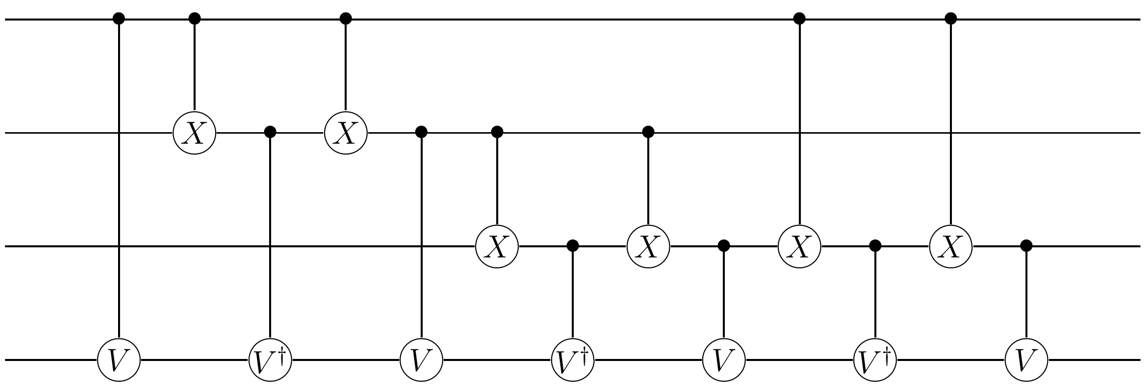
4 Further Problem

In this section we consider the case of four atoms (the system of four qubits) and present a problem on constructing the controlled–controlled–controlled NOT gate defined by



$$= \quad CCC_{NOT}$$

as a picture. This gate is usually constructed as



where V is a unitary matrix given by

$$V = \begin{pmatrix} \frac{3+i}{4} & \frac{1-i}{4} \\ \frac{1-i}{4} & \frac{3+i}{4} \end{pmatrix} \quad \Longrightarrow \quad V^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 .$$

By the way, we have given the exact form of evolution operator for the four atoms Tavis–Cummings model [7], therefore we can in principle track the same line shown in this paper and it may be possible to get the controlled–controlled–controlled NOT gate

or controlled–controlled–controlled unitary gates directly. However, such a calculation for the case becomes very complicated because we must treat 16×16 matrices at each step of calculations. We will attempt it in the near future.

It is not easy to extend our method to the general case because to obtain the exact form of evolution operator for the general Tavis–Cummings model (see (5)) is almost impossible.

By the way, in [3] we have given an idea on construction of the controlled NOT gate of type (A) by making use of the following skillful method (see Figure 3).

(i) We move the third atom from the cavity.

(ii) We insert a photon in the cavity as two atoms interact with it and subject laser fields to the atoms, and next exchange the two atoms, which gives the controlled NOT operator as shown in the preceding section.

(iii) We return the third atom (outside the cavity) to the former position.

It is easy to generalize our idea to the case of n atoms.

One of important features of our model is that n atoms are trapped **linearly** in the cavity, which has both strong points and weak points. One of weak points is that a photon inserted in the cavity interacts with all atoms, so it is impossible to select atoms which interact with the photon. To improve this we presented the idea above.@

To perform a full quantum computation we need to construct (many) controlled–controlled NOT gates or controlled–controlled unitary ones for three atoms among n–atoms, see [5]; §7. We believe that our idea is still available [4].

In principle, we can construct general quantum networks.

[4] We must estimate an influence of the "getting atoms (which are not our target) in and out" on the whole states space, which is however difficult in our model. For that we must add in (1) further terms necessary to calculate it

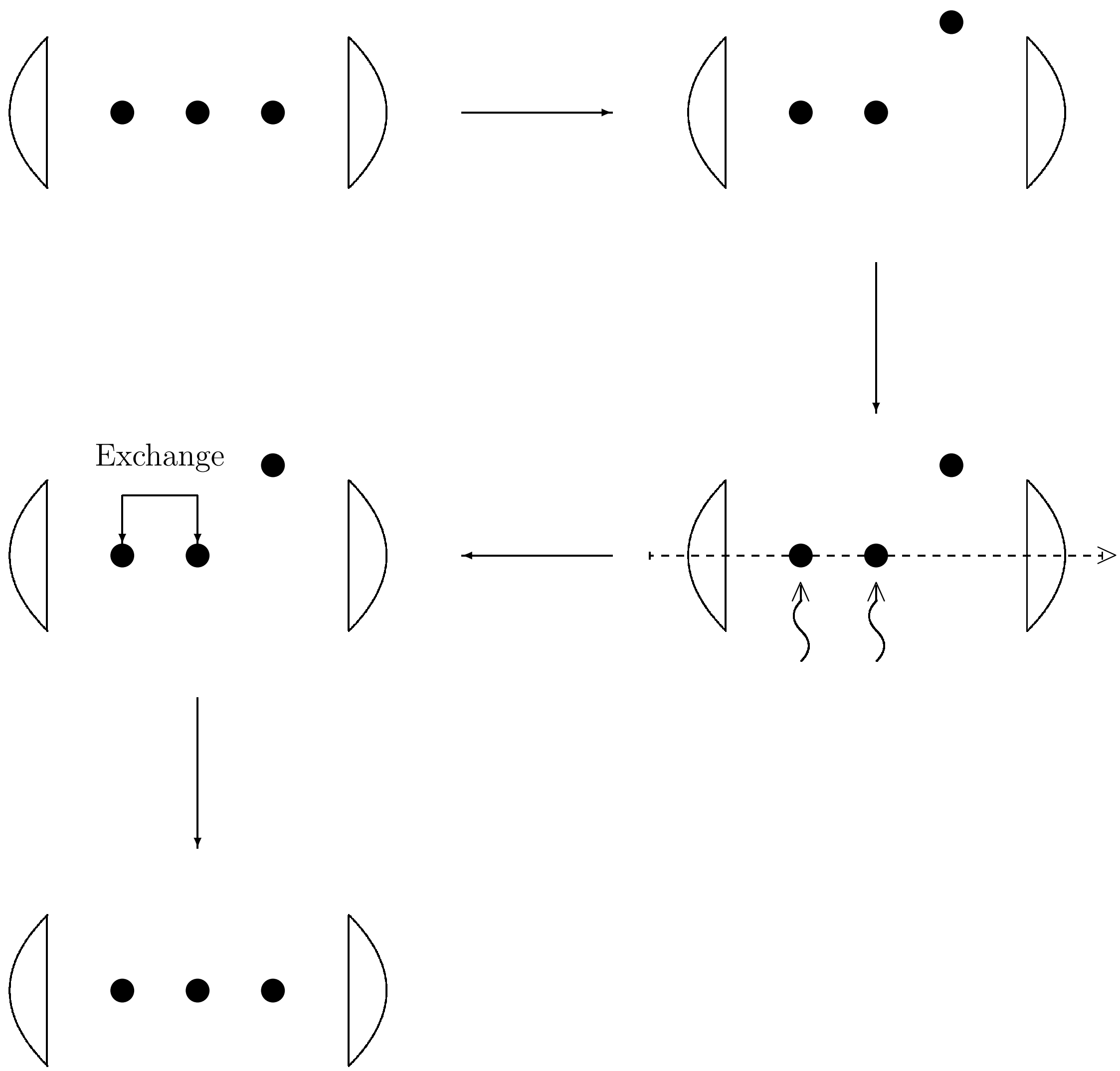


Figure 3: The process to construct the controlled NOT gate between the first atom and second one for the three atoms in a cavity

5 Discussion

In this paper we constructed the controlled NOT operator in the system of two qubits and controlled–controlled NOT operator in the system of three qubits in the quantum computation based on Cavity QED, which show that our system is universal. Therefore we can in principle perform a quantum computation. Our method is completely mathematical and we use several Rabi oscillations in a consistent manner.

We hope that some experimentalists will check whether our method works good or not.

See [13] and their references for some experiments on Cavity QED (which may be related to our method).

We conclude this paper by making a comment (which is important at least to us). The Tavis–Cummings model is based on (only) two energy levels of atoms. However, an atom has in general infinitely many energy levels, so it might be natural to use this possibility. We are also studying a quantum computation based on multi–level systems of atoms (a qudit theory) [12]. Therefore we would like to extend the Tavis–Cummings model based on two–levels to a model based on multi–levels. This is a very challenging task.

Acknowledgment. We wish to thank Tatsuo Suzuki and Shin'ichi Nojiri for their helpful comments and suggestions.

Appendix

One Qubit Operators by Classical Fields

Let us make a brief review of theory without the radiation field, whose states space is only tensor product of two level systems of each atom. See the figure 5.

The Hamiltonian in this case is

$$H=\sum_{j=1}^{n}\left\{\frac{\Delta}{2}\sigma_j^{(3)}+h_j\left(\sigma_j^{(+)}\mathrm{e}^{i(\Omega_j t+\phi_j)}+\sigma_j^{(-)}\mathrm{e}^{-i(\Omega_j t+\phi_j)}\right)\right\} \tag{59}$$

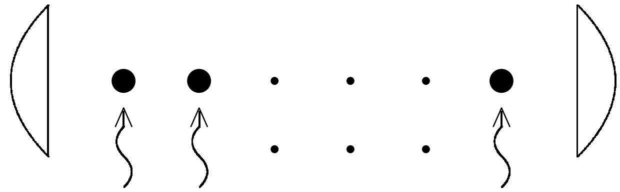


Figure 4: The n atoms in the cavity without a photon (in Figure 1)

from (1), where we have omitted the unit operator $\mathbf{1}$ for simplicity. Here let us remind the notation

$$M_j = 1_2 \otimes \cdots \otimes 1_2 \otimes M \otimes 1_2 \otimes \cdots \otimes 1_2 \quad \text{for} \quad 1 \le j \le n.$$

Then

$$\begin{aligned}
H &= \sum_{j=1}^{n} \begin{pmatrix} \frac{\Delta}{2} & h_j \mathrm{e}^{i(\Omega_j t+\phi_j)} \\ h_j \mathrm{e}^{-i(\Omega_j t+\phi_j)} & -\frac{\Delta}{2} \end{pmatrix}_j \\
&= \sum_{j=1}^{n} \left\{ \begin{pmatrix} \mathrm{e}^{i\frac{\Omega_j t+\phi_j}{2}} & \\ & \mathrm{e}^{-i\frac{\Omega_j t+\phi_j}{2}} \end{pmatrix} \begin{pmatrix} \frac{\Delta}{2} & h_j \\ h_j & -\frac{\Delta}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\frac{\Omega_j t+\phi_j}{2}} & \\ & \mathrm{e}^{i\frac{\Omega_j t+\phi_j}{2}} \end{pmatrix} \right\}_j \\
&= (U_1 \otimes \cdots \otimes U_n) \sum_{j=1}^{n} \begin{pmatrix} \frac{\Delta}{2} & h_j \\ h_j & -\frac{\Delta}{2} \end{pmatrix}_j (U_1 \otimes \cdots \otimes U_n)^{\dagger}, \qquad (60)
\end{aligned}$$

where

$$U_j = \begin{pmatrix} \mathrm{e}^{i\frac{\Omega_j t+\phi_j}{2}} & \\ & \mathrm{e}^{-i\frac{\Omega_j t+\phi_j}{2}} \end{pmatrix}.$$

The wave function defined by $i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$ with (59) can be written as a tensor product

$$|\Psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle, \qquad (61)$$

so if we define

$$|\tilde{\Psi}\rangle \equiv (U_1 \otimes \cdots \otimes U_n)^{\dagger} |\Psi\rangle,$$

then it is easy to see

$$i\frac{d}{dt}|\tilde{\Psi}\rangle = \sum_{j=1}^{n} \begin{pmatrix} \frac{\Delta-\Omega_j}{2} & h_j \\ h_j & -\frac{\Delta-\Omega_j}{2} \end{pmatrix}_j |\tilde{\Psi}\rangle.$$

Th solution is easy to obtain

$$|\tilde{\Psi}(t)\rangle = \bigotimes_{j=1}^{n} \exp\left\{-it\begin{pmatrix} \frac{\Delta-\Omega_j}{2} & h_j \\ h_j & -\frac{\Delta-\Omega_j}{2} \end{pmatrix}\right\}|\tilde{\Psi}(0)\rangle.$$

Therefore, the solution that we are looking for is

$$\begin{aligned} |\Psi(t)\rangle &= (U_1 \otimes \cdots \otimes U_n)\,|\tilde{\Psi}(t)\rangle \\ &= \bigotimes_{j=1}^{n} \begin{pmatrix} \mathrm{e}^{i\frac{\Omega_j t+\phi_j}{2}} & \\ & \mathrm{e}^{-i\frac{\Omega_j t+\phi_j}{2}} \end{pmatrix} \exp\left\{-it\begin{pmatrix} \frac{\Delta-\Omega_j}{2} & h_j \\ h_j & -\frac{\Delta-\Omega_j}{2} \end{pmatrix}\right\}|\Psi(0)\rangle. \end{aligned} \tag{62}$$

Last we note that

$$\exp\left\{-it\begin{pmatrix} \frac{\theta}{2} & h \\ h & -\frac{\theta}{2} \end{pmatrix}\right\} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

where

$$\begin{aligned} x_{11} &= \cos\left(t\sqrt{\frac{\theta^2}{4}+h^2}\right) - i\frac{\theta}{2}\frac{\sin\left(t\sqrt{\frac{\theta^2}{4}+h^2}\right)}{\sqrt{\frac{\theta^2}{4}+h^2}}, \\ x_{12} &= x_{21} = -ih\frac{\sin\left(t\sqrt{\frac{\theta^2}{4}+h^2}\right)}{\sqrt{\frac{\theta^2}{4}+h^2}}, \\ x_{22} &= \cos\left(t\sqrt{\frac{\theta^2}{4}+h^2}\right) + i\frac{\theta}{2}\frac{\sin\left(t\sqrt{\frac{\theta^2}{4}+h^2}\right)}{\sqrt{\frac{\theta^2}{4}+h^2}}. \end{aligned}$$

We can always construct unitary operators in $U(2)$ at each atom by using Rabi osillations, see for example [11].

As an example let us construct the Walsh–Hadamard operator W which has been used in the text. We set

$$V(t) = \begin{pmatrix} \mathrm{e}^{i\frac{\Omega t+\phi}{2}} & \\ & \mathrm{e}^{-i\frac{\Omega t+\phi}{2}} \end{pmatrix} \exp\left\{-it\begin{pmatrix} \frac{\Delta-\Omega}{2} & h \\ h & -\frac{\Delta-\Omega}{2} \end{pmatrix}\right\}$$

and set the resonance condition $\Delta = \Omega$, then

$$\begin{aligned} V(t) &= \begin{pmatrix} \mathrm{e}^{i\frac{\Delta t+\phi}{2}} & \\ & \mathrm{e}^{-i\frac{\Delta t+\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos(ht) & -i\sin(ht) \\ -i\sin(ht) & \cos(ht) \end{pmatrix} \\ &= \mathrm{e}^{i\frac{\Delta t+\phi}{2}} \begin{pmatrix} 1 & \\ & \mathrm{e}^{-i(\Delta t+\phi)} \end{pmatrix} \begin{pmatrix} \cos(ht) & -i\sin(ht) \\ -i\sin(ht) & \cos(ht) \end{pmatrix} \end{aligned}$$

Now we again set

$$V(t) = \begin{pmatrix} 1 & \\ & \mathrm{e}^{-i(\Delta t+\phi)} \end{pmatrix} \begin{pmatrix} \cos(ht) & -i\sin(ht) \\ -i\sin(ht) & \cos(ht) \end{pmatrix} \tag{63}$$

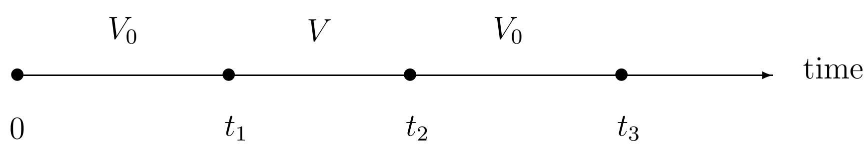
for simplicity because there is no interest in the overall phase factor [5]. On the other hand, when we don't subject a laser field to the atom ($h = 0$) we have

$$V_0(t) = \begin{pmatrix} 1 & \\ & \mathrm{e}^{-i\Delta t} \end{pmatrix}. \tag{64}$$

By using $V(t)$ and $V_0(t)$ we construct W in the following ([11]). First we set

$$V(t_f, t_i) = V(t_f - t_i), \quad V_0(t_f, t_i) = V_0(t_f - t_i) \quad \text{for} \quad t_f \ > \ t_i.$$

A sequence of operators to construct the Walsh–Hadamard gate is given as follows :



For $V_0(t_1, 0)$, $V_0(t_3, t_2)$ with $t_1 = 3\pi/2\Delta$ and $t_3 - t_2 = 3\pi/2\Delta$

$$V_0(t_1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad V_0(t_3, t_2) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and $V(t_2, t_1)$ with $t_2 - t_1 = \pi/4h$

$$V(t_2, t_1) = \begin{pmatrix} 1 & \\ & \mathrm{e}^{-i\{\Delta(t_2-t_1)+\phi\}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

we have

$$V_0(t_3, t_2) V(t_2, t_1) V_0(t_1, 0) = \begin{pmatrix} 1 & \\ & \mathrm{e}^{-i\{\Delta(t_2-t_1)+\phi\}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

[5] In the Hamiltonian (1) a constant term which makes a overall phase factor has been removed

By choosing the phase ϕ as $e^{-i\{\Delta(t_2-t_1)+\phi\}}=1$ we finally obtain

$$V_0(t_3,t_2)V(t_2,t_1)V_0(t_1,0)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=W. \tag{65}$$

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